THE ENCLOSING OF CELLS IN THREE SPACE BY SIMPLE CLOSED SURFACES

BY

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1. **Introduction.** A subset X of Euclidean n-space R^n which is homeomorphic to an Euclidean polyhedron is called *tame* by Fox-Artin [4][2] provided there is a homeomorphism of R^n onto itself which carries X onto an Euclidean polyhedron. A question that naturally arises is under what conditions will a homeomorph of an Euclidean polyhedron be tame.

As a step toward the solution of this problem Harrold [6] shows that an arc or simple closed curve in R^3 with a certain property $\mathcal{P}$ (defined below) has a complement which is homeomorphic to the complement of its prototype, an evident necessary condition for being tame.

In this paper an extension of the definition of property $\mathcal{P}$ to k-cells in R^3, $k=1$, 2, or 3, is made, and an extension of the result of Harrold to k-cells is made at the cost of imposing an extra condition which is always fulfilled when $k=1$. The techniques used are those of Harrold, and depend strongly on the results of Alexander [1]. Much use is made of the concept of a semi-linear map as used by Graeub [5] and Moise [9]. As in Harrold-Moise [7], the set K is called *locally polyhedral at a point* p provided there is a neighborhood of p which meets K in a finite (or null) polyhedron. The set K is called *locally polyhedral modulo* C if it is locally polyhedral at each point of the complement of C.

2. **Definitions and notation.** Euclidean k-space will be denoted by R^k and a fixed rectangular Cartesian coordinate system will be assumed chosen for R^k. The closure of a subset A of a space R will be denoted by $\mathrm{Cl}_R\, A$, or simply $\mathrm{Cl}\, A$ if it is clear from the context what space R is meant. The boundary of A in R will be denoted by $B_R(A)$ and is defined to be $[\mathrm{Cl}_R\, A] \cap [\mathrm{Cl}_R(R\backslash A)]$, where $X\backslash Y$ denotes the set of points in X but not in Y. The set E^k is defined to be the set of all points $(x_1, \cdots, x_k)$ of R^k such that $0 \leqq x_i \leqq 1$ for each $i=1, \cdots, k$, and C^k denotes a topological image of E^k in R^3 for $k=1$, 2, or 3. The image C^k will be called a k-cell and a 0-cell is defined to be a point. The symbol C without a superscript is to be interpreted as a k-cell for some $k=1$, 2, or 3, and the term "a cell" as "a k-cell for $k=0$, 1, 2, or 3."

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(2) The numbers in brackets refer to the bibliography.

If h is any homeomorphism of E^k onto C for $k=1$, 2, or 3, that subset of C which is the image under h of $B_{R^k}(E^k)$ is denoted by ∂C, and is seen not to depend on h. If C is a 0-cell, ∂C is defined to be the null set. An open k-cell is a homeomorph of $E^k \backslash B_{R^k}(E^k)$, and a k-sphere is a homeomorph of $B_{R^{k+1}}(E^{k+1})$. A 2-cell (an open 2-cell) will occasionally be called a disk (an open disk).

If K is a 2-sphere in R^3 then Int K and Ext K will be used to denote respectively the bounded and unbounded domains complementary to K (see [11, Theorem 5.3]).

The null set will be denoted by $\square$. The symbol $\delta(A)$ denotes the diameter of the set A, defined as usual by $\delta(A)=\sup_{a,b\in A} d(a, b)$ where $d(a, b)$ denotes Euclidean distance. Two sets A and B are called separate in R provided $A \cap \mathrm{Cl}_R B = B \cap \mathrm{Cl}_R A = \square$.

The topological product of two spaces M and N is denoted by $M \times N$. Two continuous maps f_0 and f_1 of A into B are said to be homotopic (in B) if there is a continuous map F (called a homotopy) of $A \times E^1$ into B which agrees with f_0 on $A \times 0$ and with f_1 on $A \times 1$. When f_0 and f_1 are homeomorphisms, then a homotopy F of f_0 and f_1 is called an isotopy provided the restriction of F to $A \times t$ is a homeomorphism for each t in E^1.

2.11 DEFINITION. *Let $\mathfrak{H}$ denote the non-null class of all homeomorphisms of $E^k = E^1 \times E^{k-1}$ onto C, and let*

$$\mathfrak{T}^k = \{T \mid T = h(x \times E^{k-1}) \text{ for some } x \text{ in } E^1 \text{ and } h \text{ in } \mathfrak{H}\}.$$

2.12 DEFINITION. *For each h in $\mathfrak{H}$ let*

$$\mathfrak{T}_h^k = \{T \in \mathfrak{T}^k \mid T = h(x \times E^{k-1}) \text{ for some } x \text{ in } E^1\}.$$

It is evident that if $T \in \mathfrak{T}^k$ then $C \backslash T$ is either connected or consists of exactly two components A_0 and A_1 and that $A_i \cup T = \mathrm{Cl}(A_i)$ is a k-cell, $i=0, 1$.

It is well known that the collection of all closed subsets of C forms a metric space under the Hausdorff metric [8] σ which is defined as follows.

2.13 DEFINITION. $\sigma(A, B) = \max\,[\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)]$.

By the remark following Definition 2.12 if $T \in \mathfrak{T}^k$ then $C \backslash T$ may be written as $M' \cup N'$, where M' is a component of $C \backslash T$ and N' is either a component of $C \backslash T$ or the null set. Letting $M = M' \cup T$ and $N = N' \cup T$, T determines a triple of non-null closed subsets (T, M, N) of C. For any pair T_1, T_2 of elements of $\mathfrak{T}^k$, make the following definition.

2.14 DEFINITION.

$$\rho(T_1, T_2) = \sigma(T_1, T_2) + \min\,[\sigma(M_1, M_2) + \sigma(N_1, N_2), \sigma(M_1, N_2) + \sigma(N_1, M_2)].$$

Evidently $\rho(T, T)=0$, and if $\rho(T_1, T_2)=0$ then $\sigma(T_1, T_2)=0$ so that $T_1 = T_2$. For any triple T_1, T_2, T_3 of elements of $\mathfrak{T}^k$ and a proper choice of notation

$$\rho(T_1, T_2) = \sigma(T_1, T_2) + \sigma(M_1, M_2) + \sigma(N_1, N_2),$$

and

$$\rho(T_2, T_3) = \sigma(T_2, T_3) + \sigma(M_2, M_3) + \sigma(N_2, N_3).$$

Adding these two and using the triangle inequality for σ gives

$$\rho(T_1, T_2) + \rho(T_2, T_3) \geqq \sigma(T_1, T_3) + \sigma(M_1, M_3) + \sigma(N_1, N_3) \geqq \rho(T_1, T_3).$$

Thus ρ is a metric on $\mathfrak{T}^k$.

The superscript of $\mathfrak{T}^k$ will hereafter be omitted when no loss of clarity results.

2.21 Definition. *For every $T\in\mathfrak{T}$ and $\epsilon>0$ define $\mathfrak{P}(T, \epsilon)$ as the set of all $K\subset R^3$ satisfying the following conditions:*

1. *K is a topological 2-sphere.*
2. *$T\subset \operatorname{Int} K$.*
3. *K is locally polyhedral modulo C.*
4. *$K\cap C=T_1\cup T_2$, where $T_1\cap T_2=\square$, where $T_i=\square$ or $T_i\in\mathfrak{T}$ for $i=1, 2$, and where if $T\subset\partial C$, then $T_2=\square$.*
5. *$K\subset S(T, \epsilon)$.*

It is evident that both $\mathfrak{T}$ and $\mathfrak{P}(T, \epsilon)$ depend on the cell C. Since with few exceptions only one cell C will be under consideration, this dependence is not indicated in the notation. However, when this is not the case, these sets will be denoted by $\mathfrak{T}(C)$ and $\mathfrak{P}(C, T, \epsilon)$ respectively.

2.22 Definition. *C will be said to have property $\mathcal{P}$ provided that for each $T\in\mathfrak{T}$ and $\epsilon>0$ the set $\mathfrak{P}(T, \epsilon)$ is non-null.*

2.23 Definition. *C will be said to have property $\mathcal{P}$ relative to a subset $\mathfrak{T}_0$ of $\mathfrak{T}$ provided for every $\epsilon>0$ and $T\in\mathfrak{T}_0$ the set $\mathfrak{P}(T, \epsilon)$ is non-null.*

2.31 Definition. *For every $T\in\mathfrak{T}$ and $\epsilon>0$ let $\mathfrak{D}(T, \epsilon)$ denote the set of all $D\subset R^3$ such that*

1. *D is a topological 2-cell.*
2. *$\partial D\cap C=\square$.*
3. *$D\cap C\in\mathfrak{T}$.*
4. *D is locally polyhedral modulo C.*
5. *$\rho(T, D\cap C)<\epsilon$.*
6. *If $C\backslash D$ has two components C_1 and C_2, then there is an $\eta>0$ such that if N is a connected set meeting both C_1 and C_2 with $\delta(N)<\eta$, then N meets D also.*

2.32 Definition. *C will be said to have the disk property relative to a subset $\mathfrak{T}_0$ of $\mathfrak{T}$ provided $\mathfrak{D}(T, \epsilon)$ is non-null for every $\epsilon>0$ and $T\in\mathfrak{T}_0$. If C has the disk property relative to the whole set $\mathfrak{T}$, then it will simply be said that C has the disk property.*

2.33 Definition. *C will be said to have the uniform disk property relative to a subset $\mathfrak{T}_0$ of $\mathfrak{T}$ provided to each $\omega>0$ there corresponds a $\delta>0$ such that if $T\in\mathfrak{T}_0$ and $\epsilon>0$ there is a D in $\mathfrak{D}(T, \epsilon)$ with $d(\partial D, C)>\delta$ and $D\subset S(T, \omega)$.*

When C has the uniform disk property relative to $\mathfrak{T}_h$ for each h in $\mathfrak{H}$, C will be said to have the uniform disk property.

2.4 DEFINITION. *C will be said to have the enclosure property provided for each $\epsilon>0$ there is a polyhedral topological 2-sphere K in $S(C, \epsilon)$ with $C\subset \operatorname{Int} K$.*

3. **Relations between the metrics.** The metric ρ is chosen for $\mathfrak{T}$ in preference to the somewhat more simple metric σ chiefly because σ does not have the property described in this lemma.

3.1 LEMMA. *If U_1, U_2, and T are elements of $\mathfrak{T}$ such that T separates U_1 and U_2 on C, then there is a $\beta>0$ such that every element T' of $\mathfrak{T}$ with $\rho(T, T')<\beta$ separates U_1 and U_2 on C.*

Proof. The notation may be assumed chosen so that $M_0\supset U_1$ and $N_0\supset U_2$ where M_0 and N_0 are the components of $C\backslash T$. If $T'\in\mathfrak{T}$ is disjoint from U_1 and U_2 and does not separate U_1 and U_2 on C, then both U_1 and U_2 lie in the same component M_0' of $C\backslash T'$. Choose $p_i\in U_i$, $i=1, 2$ and let $M=M_0\cup T$, $N=N_0\cup T$, $M'=M_0'\cup T'$, and $N'=N_0'\cup T'$ where N_0' is either null or the component of $C\backslash T'$ other than M_0'. Then

$$\rho(T, T') \geqq \min\,[\sigma(M, M') + \sigma(N, N'), \sigma(M, N') + \sigma(N, M')],$$

so

$$\rho(T, T') \geqq \min\,[\sigma(M, M'), \sigma(N, M')],$$

or

$$\rho(T, T') \geqq \min\left[\sup_{p\in M'} d(M, p), \sup_{p\in M'} d(N, p)\right];$$

or

$$\rho(T, T') \geqq \min\,[d(M, p_2), d(N, p_1)].$$

But $p_2\in U_2\subset N$ while $p_1\in U_1\subset M$, so $\beta_1=\min\,[d(M, p_2), d(N, p_1)]>0$. Also $T\cap(U_1\cup U_2)=\square$, so $\beta_2=d(T, U_1\cup U_2)>0$. Hence $\beta=\min\,(\beta_1, \beta_2)$ is positive and independent of T'. Further, if $\rho(T, T')<\beta$, then the assumption that T' does not separate U_1 and U_2 on C leads to a contradiction.

3.2. LEMMA. *If $T\in\mathfrak{T}$ and $\delta>0$ then there is an $\eta>0$ such that the complement of $S(T, \delta)$ lies in* Ext *K for every K in $\mathfrak{P}(T, \eta)$.*

Proof. Choose $p\in R^3$ and r sufficiently large that $S(T, \delta)\subset S(p, r)$. Then $R=R^3\backslash S(p, r)$ is connected and disjoint from $S(T, \delta/n)$ for each $n=1, 2, \cdots$, and hence determines a component R_n of $R^3\backslash S\,(T, \delta/n)$ for each n. The sequence $\{R_n\}$ is monotone nondecreasing and has as limit $\bigcup R_n$, which is clearly in $R^3\backslash T$. But any point x of $R^3\backslash T$ can be joined to R by an arc A_x in $R^3\backslash T$ and if δ/n is less than $d(T, A_x\cup R)$ then A_x and hence x is in R_n. Thus the reverse inclusion $(R^3\backslash T)\subset\bigcup R_n$ also holds and $\bigcup R_n=R^3\backslash T$. This means that any closed set disjoint from T lies in R_n for all sufficiently large n. In particular, for the assigned δ there is an N such that the complement of $S(T, \delta)$ lies in R_N. Now if $\eta<d(T, R_N)$ and $K\in P(T, \eta)$, then R_N is an unbounded set in the complement of K and hence $R_N\subset\text{Ext } K$. Since the complement of $S(T, \delta)$ lies in R_N, the desired η has been found.

3.3 COROLLARY. *A cell has property $\mathcal{P}$ (property $\mathcal{P}$ relative to $\mathfrak{T}_0$) if and only if for each $\epsilon>0$ and $T\in\mathfrak{T}$ ($T\in\mathfrak{T}_0$) there is a K in $\mathfrak{P}(T, \epsilon)$ with* Int $K\subset S(T, \epsilon)$.

4. Property $\mathcal{P}$ and the disk property.

4.1. THEOREM. *In order that C have property $\mathcal{P}$ (property $\mathcal{P}$ relative to $\mathfrak{T}_0$) it is necessary and sufficient that for every $\epsilon>0$ and $T\in\mathfrak{T}$ ($T\in\mathfrak{T}_0$) there be a K in $\mathfrak{P}(T, \epsilon)$ satisfying the following two conditions:*

1. $(K\cup\text{Int } K)\subset S(T, \epsilon)$.
2. *If $K\cap C$ has two components, U_1 and U_2, then T separates U_1 and U_2 on C.*

Proof. The sufficiency is obvious, so suppose C has property $\mathcal{P}$ (property $\mathcal{P}$ relative to $\mathfrak{T}_0$) and let $\epsilon>0$ and $T\in\mathfrak{T}$ ($T\in\mathfrak{T}_0$) be assigned. Two cases arise.

CASE 1. $T\cap\partial C=\partial T$. Then $T=h(a\times E^{k-1})$ for some $h\in\mathfrak{H}$ and $0<a<1$, and $M_1=\bigcup_{0\leq x<a} h(x\times E^{k-1})$ and $M_2=\bigcup_{a<x\leq 1} h(x\times E^{k-1})$ are the components of $C\backslash T$. Let p_i be a point of M_i, $i=1, 2$, and let $\eta=\min\,[\epsilon, d(T, p_1), d(T, p_2)]$. Then $\mathfrak{P}(T, \eta)\subset\mathfrak{P}(T, \epsilon)$ since $\eta<\epsilon$, so it will suffice to show there is a K in $\mathfrak{P}(T, \eta)$ meeting the requirements. Corollary 3.3 guarantees there is a K in $\mathfrak{P}(T, \eta)$ with Int $K\subset S(T, \eta)$, and this, together with 5 of Definition 2.21 means that this K satisfies condition 1.

Now let i be either 1 or 2. Since $d(p_i, T)\geqq\eta$ and $(K\cup\text{Int } K)\subset S(T, \eta)$, p_i is in Ext K. So Cl $M_i=M_i\cup T$ contains $p_i\in\text{Ext } K$ and $T\subset\text{Int } K$, so $K\cap\text{Cl } M_i=K\cap(M_i\cup T)=K\cap M_i\neq\square$. Thus $K\cap M_i=U_i\in\mathfrak{T}$ by 4 of Definition 2.21. But since T separates M_1 and M_2 on C, it must also separate $U_1=M_1\cap K$ and $U_2=M_2\cap K$ on C, so condition 2 is satisfied.

CASE 2. $T\subset\partial C$. Then Corollary 3.3 guarantees condition 1 and condition 2 is vacuously fulfilled, for condition 4 of Definition 2.21 requires that $K\cap C$ have but one component.

4.2. DEFINITION. *Let $\mathfrak{P}^*(T, \epsilon)$ denote the set of all spheres of $\mathfrak{P}(T, \epsilon)$ satisfying the conclusion of Theorem* 4.1.

In terms of this notation Theorem 4.1 states that C has property $\mathcal{P}$ (relative to $\mathfrak{T}_0$) if and only if $\mathfrak{P}^*(T, \epsilon)$ is non-null for every $T\in\mathfrak{T}$ ($T\in\mathfrak{T}_0$) and $\epsilon>0$.

4.3. LEMMA. *If $T\in\mathfrak{T}$ and $\epsilon>0$, then there is a $\beta>0$ such that whenever $K\in\mathfrak{P}^*(T, \beta)$ and T_1 is a component of $K\cap C$, then $\rho(T, T_1)<\epsilon$.*

Proof. Let $\epsilon > 0$ and $T \in \mathfrak{T}$ be assigned. Then C may be written as $C_1 \cup C_2$ where $T = h(x_0 \times E^{k-1})$ for some $h \in \mathfrak{H}$ and C_1 and C_2 are the images under h of $\bigcup_{0 \leq x \leq x_0} h(x \times E^{k-1})$ and $\bigcup_{x_0 \leq x \leq 1} h(x \times E^{k-1})$ respectively. Each C_i is a cell (one is of dimension k and the other of dimension k or $k-1$ according as $T \cap \partial C$ is ∂T or T) and is therefore uniformly locally connected. That is, for $i=1$ and $i=2$ there is an $\omega_i > 0$ such that if $t \in T$ and $p \in C_i$ with $d(t, p) < \omega_i$ then some connected subset Q of C_i with $\delta(Q) < \epsilon/4$ contains both t and p.

If $C_i \neq T$, straightforward application of the uniform continuity of h produces an $x_i \neq x_0$ meeting the following conditions:

4.31 $T_i' = h(x_i \times E^{k-1})$ is a subset of C_i.

4.32 $d(t, T_i') < \omega_i$ for all $t \in T$.

4.33 $d[h(x, y), T] + d[h(x, y), T_i'] < \epsilon/4$ for all x between x_i and x_0 and any y in E^{k-1}.

For $i = 1, 2$ let $\beta_i = \min\,[d(T, T_i'), \epsilon/4]$ or $\beta_i = 1$ according as $C_i \neq T$ or $C_i = T$, and let $\beta = \min\,(\beta_1, \beta_2)$.

Suppose now that $K \in \mathfrak{P}^*(T, \beta)$ and that T_1 is a component of $K \cap C$. The inclusion $T_1 \subset C_1$ is assumed as a notational convenience. If t is any point of T then by 4.32 there is a $t_1' \in T_1'$ with $d(t, t_1') < \omega_1$ and hence a connected subset Q of C_1 with $\delta(Q) < \epsilon/4$ which contains both t and t_1'. But $T \subset \text{Int } K$ and, since $K \cup \text{Int } K \subset S(T, \beta)$ and $d(T, T_1') > \beta$, $T_1' \subset \text{Ext } K$. The set Q therefore meets both Int K and Ext K, and must meet K, so $t_1 \in Q \cap K$ may be chosen. That $K \cap C_1 = T_1$ follows from condition 2 of Theorem 4.1 and the choice of the sets C_1 and C_2, so since $Q \cap K \subset C_1 \cap K$, t_1 is a point of T_1. Then $d(t, T_1) \leq d(t, t_1) \leq \delta(Q) \leq \epsilon/4$. But t was arbitrary in T so $\sup_{t \in T} d(t, T_1) \leq \epsilon/4$. Since $T_1 \subset K \subset S(T, \beta)$, it follows that $\sup_{t_1 \in T_1} d(T, t_1) < \beta \leq \epsilon/4$. Combining these yields $\sigma(T, T_1) \leq \epsilon/4$.

Now let (T, C_1, C_2) and (T_1, M, N) be the triples of sets in the expression for $\rho(T, T_1)$ where N contains T. Since C_2 is connected and $T_1 \subset C_1$, this requires $C_2 \subset N$ and $M \subset C_1$, so that $\sup_{p \in M} d(p, C_1) = \sup_{q \in C_2} d(N, q) = 0$. This means that

$$\rho(T, T_1) \leq \sigma(T, T_1) + \sup_{p \in C_1} d(M, p) + \sup_{q \in N} d(q, C_2).$$

But if $p \in C_1$, then $p = h(x, y)$ where $0 \leq x \leq x_0$ and $y \in E^{k-1}$. If $x < x_1$ then $p \in M$ so $d(M, p) = 0$, and if $x_1 \leq x \leq x_0$, then, by 4.33 together with the fact that the choice of β implies $T_1' \subset M$, $d(M, p) \leq d(T_1', p) \leq \epsilon/4$. Similarly, if $q \in N$ then $q = h(x, y)$ with $x_1 \leq x \leq 1$ so either $x_0 \leq x$, in which case $q \in C_2$ and $d(q, C_2) = 0$, or $x_1 \leq x \leq x_0$, so that by 4.33, $d(q, C_2) \leq d(q, T) < \epsilon/4$. Combining, $\rho(T, T_1) \leq \epsilon/4 + \epsilon/4 + \epsilon/4 < \epsilon$.

4.4. THEOREM. *If C has property $\mathfrak{P}$ (relative to $\mathfrak{T}_h$), then C has the disk property (relative to $\mathfrak{T}_h$).*

Proof. Let $T \in \mathfrak{T}$ ($T \in \mathfrak{T}_h$) and $\epsilon > 0$ be assigned. Then $C \setminus T$ may be written as $N_1 \cup N_2$, where N_1 is a component and N_2 is either a component or null.

Since $T=h(x\times E^{k-1})$ for some $h\in\mathfrak{H}$ (the given h) and some x $N_1\cup T$ is a k-cell. Choose $p_1\in N_1$ and let $\beta_1=d(p_1, T)$. By Lemma 4.3 there is a $\beta_2>0$ such that if $K\in\mathfrak{P}^*(T, \beta_2)$ and T_1 is a component of $K\cap C$, then $\rho(T, T_1)<\epsilon$. Let $\beta=\min(\beta_1, \beta_2)$ and choose $K\in\mathfrak{P}^*(T, \beta)$.

Now $K\cap C$ has either a single component T_1 or two components T_1 and T_2 according as N_2 is null or not, and this together with the choice of β and condition 1 of Theorem 4.1 guarantees that $K\cap M_1\neq\square$ so that $K\cap M_1=T_1\in\mathfrak{T}$ is but a notational convenience.

If $T_2\neq\square$, then $T_2\in\mathfrak{T}$. K is then a topological 2-sphere with T_1 and T_2 a pair of disjoint $(k-1)$-cells on K. K is, then, the image of the unit spherical surface W under a homeomorphism g which maps V onto T_1 where V is the point $(1, 0, 0)$, or the set $\{(x, y, 0)\in R^3 \mid x^2+y^2=1 \text{ and } x\geqq 0\}$, or the set $\{(x, y, z)\in R^3 \mid x^2+y^2+z^2=1 \text{ and } x\geqq 0\}$, according as C is a 1-cell, a 2-cell, or a 3-cell. The set $g^{-1}(T_2)$ is disjoint from V so for some $\alpha>0$, $\{\text{Cl }[W\cap S(V, \alpha)]\}\cap\{g^{-1}(T_2)\}=\square$. If $X=W\cap S(V, \alpha)$, then ∂X is a simple closed curve separating $g^{-1}(T_2)$ and $V=g^{-1}(T_1)$ on W. Then $S=g(\partial X)$ is a simple closed curve separating T_1 and T_2 on K.

If T_2 is null S may be chosen to be any simple closed curve on $K\backslash C$. In either case then, the curve S divides K into two closed disks, one of which meets C at T_1 and at no other points. Call this disk D. It will now be shown that D, T, and ϵ have the six properties of Definition 2.31.

1. D is a topological 2-cell by the Schoenflies theorem.
2. $\partial D=S$ and $S\subset K\backslash C$ so $\partial D\cap C=\square$.
3. $D\cap C=T_1\in\mathfrak{T}$.
4. $D\subset K$ so D is locally polyhedral mod C by 3 of Definition 2.21.
5. $\rho(T, D\cap C)=\rho(T, T_1)<\epsilon$ by choice of β and K.
6. Since $K\cap N_1=T_1$ and K separates $p\in\text{Ext } K$ from $T\subset\text{Int } K$ on $N_1\cup T$, $(N_1\cup T)\backslash T_1$ may be written as the union of $C_1=N_1\cap\text{Ext } K$ containing p and $C_2=(N_1\cup T)\cap\text{Int } K$ containing T. Since $(\text{Cl } N_1)\cap(\text{Cl } N_2)$ is null or T according as N_2 is null or not, it follows that C_1 and N_2 are separate so that $T_1\cap\text{Cl } N_2=\square$ and the two components of $C\backslash T_1$ must be C_1 and $C_2\cup N_2$. Thus $\omega>0$ can be taken less than $d[T_1, N_2\cup(K\backslash D)]$ so that $N_2\cap S(T_1, \omega)=\square$ and $K\cap S(T_1, \omega)=D\cap S(T_1, \omega)$.

It is asserted that there is then an $\eta>0$ such that if M is a connected set meeting both C_1 and $C_2\cup N_2$ with $\delta(M)<\eta$, then $M\subset S(T_1, \omega)$. For if this is not the case for $i=1, 2, \cdots$ there is a connected set M_i with $\delta(M_i)<1/i$ and a triple of points, $p_i\in M_i\cap C_1$, $q_i\in M_i\cap(C_2\cup N_2)$ and $r_i\in M_i$ with $d(r_i, T_1)>\omega$. By a standard procedure a subsequence of indices k_1, $k_2, \cdots$ can be chosen such that $\{p_{k_i}\}$, $\{q_{k_i}\}$ and $\{r_{k_i}\}$ all converge, and it is readily seen that they must all have the same limit point q. As the limit of $\{p_{k_i}\}\subset C_1$ and $\{q_{k_i}\}\subset C_2\cup N_2$, q must lie in $(\text{Cl } C_1)\cap\text{Cl }(C_2\cup N_2)=T_1$, contradicting the fact that as the limit of $\{r_{k_i}\}$, $d(q, T_1)\geqq\omega$. Hence the asserted η exists, and is the number required to fulfil condition 6. For suppose M is connected, $\delta(M) < \eta$, and M meets both C_1 and $C_2 \cup N_2$. Then $M \subset S(T_1, \omega)$ so $M \cap N_2 \subset S(T_1, \omega) \cap N_2 = \square$ and M meets both $C_1 \subset \text{Ext } K$ and $C_2 \subset \text{Int } K$. This requires that M meet K and $M \cap K \subset K \cap S(T_1, \omega) = D \cap S(T_1, \omega)$.

Thus $D \in \mathfrak{D}(T, \epsilon) \neq \square$, and C has the disk property (relative to $\mathfrak{T}_h$).

5. **The interior radii.**

5.1. THEOREM. *If $\mathfrak{T}_0$ is any compact subset of $\mathfrak{T}$ and C has property $\mathcal{P}$ relative to $\mathfrak{T}_0$, then for each $\epsilon > 0$ there is a $\gamma > 0$ such that for every $T \in \mathfrak{T}_0$ the inequality* $\sup_{K \in \mathfrak{P}^*(T, \epsilon)} d(T, K) > \gamma$ *holds.*

Proof. Let $\gamma(T, \epsilon)$ denote the supremum in the statement of the theorem. If the conclusion is false, then for each integer n there is a T_n in $\mathfrak{T}_0$ such that $\gamma(T_n, \epsilon) \leqq 1/n$. Since $\mathfrak{T}_0$ is compact, some subsequence $\{T_{n_i}\}$ converges to $T_0 \in \mathfrak{T}_0$, and to simplify the notation it will be assumed that $\{T_n\}$ converges to T_0. Now $\mathfrak{P}^*(T_0, \epsilon/2)$ is non-null from the hypothesis that C has property $\mathcal{P}$ relative to $\mathfrak{T}_0$ and Theorem 4.1, so choose $K \in \mathfrak{P}^*(T_0, \epsilon/2)$.

For any fixed i there is a $p \in T_i$ and a $q \in K$ such that

$$d(p, q) = d(T_i, K). \tag{5.11}$$

So for any $r \in T_0$, $d(p, q) + d(p, r) \geqq d(q, r)$, or

$$d(T_i, K) \geqq d(q, r) - d(p, r). \tag{5.12}$$

Since $T_0 \subset \text{Int } K$, $d(T_0, K) = 2\omega > 0$, and $d(q, r) \geqq d(T_0, K)$, so 5.12 yields

$$d(T_i, K) \geqq 2\omega - d(p, r). \tag{5.13}$$

But r was arbitrary in T_0, so

$$d(T_i, K) \geqq 2\omega - \inf_{r \in T_0} d(p, r) = 2\omega - d(p, T_0),$$

and, since $\rho(T_i, T_0) \geqq d(p, T_0)$, $d(T_i, K) \geqq 2\omega - \rho(T_i, T_0)$. But $\{T_i\}$ converges to T_0 so there is an M_1 such that for all $i > M_1$, $\rho(T_i, T_0) < \omega$. Combining these facts,

$$i > M_1 \quad \text{implies} \quad d(T_i, K) \geqq 2\omega - \omega = \omega. \tag{5.14}$$

Now if $x \in R^3$ is such that $d(x, T_0) < \omega$, since $d(T_0, K) = 2\omega$, then $x \in \text{Int } K$. But if $i > M_1$, $\rho(T_i, T_0) < \omega$ so $d(x, T_0) < \omega$ for every x in T_i, i.e.,

$$i > M_1 \quad \text{implies} \quad T_i \subset \text{Int } K. \tag{5.15}$$

Let q be any point of $K \cup \text{Int } K$. Since $K \cup \text{Int } K \subset S(T_0, \epsilon/2)$, there is a point r of T_0 such that $d(r, q) < \epsilon/2$. Corresponding to this r there is a $p_i \in T_i$ such that $d(r, p_i) = d(r, T_i) \leqq \rho(T_0, T_i)$. Then $d(p_i, q) \leqq d(p_i, r) + d(r, q) \leqq \rho(T_0, T_i) + \epsilon/2$. But there is an M_2 such that if $i > M_2$ then $\rho(T_0, T_i) < \epsilon/2$. For such i the above inequality is $d(r, p_i) \leqq \epsilon/2 + \epsilon/2 = \epsilon$. So, given q arbitrary in $K \cup \text{Int } K$ and $i > M_2$ there is a $p_i \in T_i$ such that $d(q, p_i) < \epsilon$. The point p_i

depends on q, but the M_2 depends only on the convergence of $\{T_i\}$ to T_0. So

$$(5.16)\qquad i > M_2 \text{ implies } K \cup \operatorname{Int} K \subset S(T_i, \epsilon).$$

Suppose now that $K \cap C = U_1 \cup U_2$ where U_1 and U_2 are in $\mathfrak{T}$. Then since $K \in P^*(\mathfrak{T}_0, \epsilon/2)$, condition 2 of Theorem 4.1 requires that T_0 separate U_1 and U_2 on C. By Lemma 3.1, if $\rho(T_i, T_0)$ is less than some fixed $\beta > 0$, T_i also separates U_1 and U_2 on C, so there is an M_3 such that this is the case for all $i > M_3$. If $K \cap C$ does not have two components take $M_3 = 1$. In any case then

$$(5.17)\qquad \begin{array}{c} i > M_3 \text{ implies when } K \cap C \text{ has two components} \\ \text{they are separated on } C \text{ by } T_i. \end{array}$$

Now let $M = \max(M_1, M_2, M_3)$, and suppose $i > M$. Then K satisfies conditions 1, 3, and 4 of Definition 2.21 for the set $\mathfrak{P}(T_i, \epsilon)$ by virtue of being in $\mathfrak{P}^*(T_0, \epsilon/2)$, since these conditions do not involve the T and ϵ. But 5.15 and 5.16 are valid since $i > M$, so conditions 2 and 5 of Definition 2.21 are also satisfied and $K \in \mathfrak{P}(T_i, \epsilon)$. Conditions 1 and 2 of Theorem 4.1 are both fulfilled by K, T_i, and ϵ by virtue of 5.16 and 5.17. Thus $i > M$ implies $K \in \mathfrak{P}^*(T_i, \epsilon)$.

But then $\gamma(T_i, \epsilon) \geqq d(T_i, K)$, so from 5.14, $i > M$ implies $\gamma(T_i, \epsilon) > \omega$. Since $\gamma(T_i, \epsilon) < 1/i$ by choice, this is a contradiction and the theorem is established.

5.2. Theorem. *$\mathfrak{T}_h$ is an arc in $\mathfrak{T}$ for every h in $\mathfrak{H}$.*

Proof. Define $\phi: E^1 \to \mathfrak{T}_h$ bv

$$\phi(x) = h(x \times E^{k-1}) \equiv_d T_x.$$

It is obvious that ϕ is 1-1 and onto. Let $x_1 \in E^1$ and $\omega > 0$ be assigned. Since h is uniformly continuous, there is a $\theta > 0$ such that whenever $|x_2 - x_1| < \theta$, then $d[h(x_1, y), h(x_2, y)] < \omega/3$ for all y in E^{k-1}. Let M_i be the set $\{h(x, y) \mid 0 \leqq x \leqq x_i\}$, and N_i be $\{h(x, y) \mid x_i \leqq x \leqq 1\}$, $i = 1, 2$. Then $\rho(T_1, T_2) = \sigma(T_1, T_2) + \sigma(M_1, M_2) + \sigma(N_1, N_2)$, and an elementary calculation shows each of the terms on the right is less than $\omega/3$. Thus ϕ is a continuous 1-1 map of a compact space onto a Hausdorff space and must be topological, so $\mathfrak{T}_h$ is an arc.

5.3. Corollary. *If C has property $\mathcal{P}$ relative to $\mathfrak{T}_h$ and T_x denotes the $(k-1)$-cell $h(x \times E^{k-1})$, then for each $\epsilon > 0$ there is a $\gamma > 0$ such that for all x in E^1 the inequality $\gamma(T_x, \epsilon) > \gamma$ holds.*

6. **The construction lemmas.** Let h be a fixed homeomorphism of E^k onto C and let a be a point of $T_0 \setminus \partial T_0$, where T_x denotes $h(x \times E^{k-1})$, $0 \leqq x \leqq 1$. Since $a \in \operatorname{Cl}(R^3 \setminus C)$, well known results in the theory of accessibility assure that there is an arc A' from a point a_1 to a which meets C only at a. If A'' is a topological ray in $R^3 \setminus C$ with initial point a_1, then $A' \cup A''$ contains a topological ray A which meets C only at its initial point a.

Similarly a topological ray B meeting C only at its initial point $b \in T_1 \backslash \partial T_1$ can be chosen, and the two rays A and B may be taken disjoint and locally polyhedral modulo C.

A partial order on the elements of $\mathfrak{T}$ is now defined by $T < U$ provided both the conditions T separates A and U on $A \cup C \cup B$ and U separates T and B on $A \cup C \cup B$ are met. This order is extended to the set consisting of all elements of T and all points of $A \cup B$ by $U < T$ and $T < V$ for every choice of $U \in A$, $V \in B$, and T an element of $\mathfrak{T}$ which separates A and B on $A \cup C \cup B$. For each T in the set on which this order is defined it will be convenient to let $\mathcal{A}(T)$ and $\mathcal{B}(T)$ denote the components of $(A \cup C \cup B) \backslash T$ containing an unbounded subset of A and an unbounded subset of B respectively. It is easily verified that $T < U$ holds if and only if both $T \cup \mathcal{A}(T) \subset \mathcal{A}(U)$ and $\mathcal{B}(T) \supset U \cup \mathcal{B}(U)$.

It should also be noted that for elements of $\mathfrak{T}$ the order relation depends only on the choice of $a \in T_0 \backslash \partial T_0$ and $b \in T_1 \backslash \partial T_1$, and not on the choice of rays A, B, from a, b. In the remainder of this section h, A, and B will be assumed chosen and fixed.

6.1. Lemma. *Let D, K_1, T and $\epsilon > 0$ be related as follows:*

1. *$T \in \mathfrak{T}$ and separates A and B on $A \cup C \cup B$.*
2. *$K \in \mathfrak{P}^*(T, \epsilon)$.*
3. *$K \cap C = U_1 \cup U_2$ where both U_1 and U_2 are in $\mathfrak{T}$.*
4. *Either $U_1 < T < U < U_2$ or $U_1 < U < T < U_2$, where $D \cap C = U \in \mathfrak{T}$.*
5. *$D \in \mathfrak{D}(U, \omega)$ for every $\omega > 0$.*
6. *$(D \cup K_1) \cap (A \cup B) = \square$.*
7. *$d(\partial D, C) > \epsilon$.*

Then there is a K_2 in $\mathfrak{P}^*(T, \epsilon)$ with $K_2 \cap C$ either $U_1 \cup U$ or $U \cup U_2$ according as $T < U$ or $U < T$.

Proof. Suppose $U_1 < T < U < U_2$. Then these four sets are pairwise disjoint and U separates U_1 and U_2 on C. Since $U \in \mathfrak{T}$, there is a $g \in \mathfrak{H}$ such that $U = g(x \times E^{k-1})$ for some x with $0 < x < 1$, for neither $g(0 \times E^{k-1})$ nor $g(1 \times E^{k-1})$ can separate U_1 and U_2. Consequently $C \backslash U = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A}$ contains U_1 and T, $\mathcal{B}$ contains U_2, and $\mathcal{A} \cup U$ is a k-cell. An arc E in $\mathcal{A} \cup U$ with one endpoint in U, the other in U_1 and otherwise disjoint from $U \cup U_1$ can be constructed. Since T separates U_1 and U on C, E meets T so a sub-arc E' of E meeting both T and U and not meeting U_1 can be found. Then $T \cup U \cup E'$ is a connected set in the complement of K_1 so since $T \subset \text{Int } K_1$, U also is in Int K_1.

Thus $D \cap \text{Int } K_1 \neq \square$, and since $K_1 \cup \text{Int } K_1 \subset S(T, \epsilon) \subset S(C, \epsilon)$ while $d(\partial D, C) > \epsilon$, $D \cap \text{Ext } K_1 \neq \square$ so that $K_1 \cap D \neq \square$. Since $D \cap C \cap K_1 = U \cap K_1$ and $U \subset \text{Int } K_1$, $D \cap C \cap K_1 = \square$ so both D and K_1 are locally polyhedral in some neighborhood V of $D \cap K_1$ containing no points of $C \cup \partial D$, and V may be taken disjoint from $A \cup B$ also by condition 6 of the hypothesis. By shifting the vertices of D which lie in V a distance so small that no point of D outside V is moved, D and K_1 may be brought into relative general position. Then $D\cap K_1$ will consist of a finite collection of mutually disjoint simple closed curves $s_1, s_2, \cdots, s_n$.

Each s_k bounds a unique sub-disk D_k of D and a pair of disks X_k and Y_k on K_1, where Y_k contains U_1. An arc c of C can be chosen so that c meets ∂C only at its two end points a and b. Then $A\cup c\cup B$ is a topological line and hence a continuous 1-cycle, so the theory of linkages may be applied to $A\cup c\cup B$ and the curves s_k. The arc c may be chosen so that it meets each of the sets U_1, U, and U_2 in a single point, so that for any choice of k either $X_k\cap C=U_2$, $Y_k\cap C=U_1$ and $D_k\cap C=U$ or $X_k\cap C=\square$, $Y_k\cap C=U_1\cup U_2$, and $D_k\cap C=\square$ according as s_k links $A\cup c\cup B$ or not. Thus if any s_k fails to link $A\cup c\cup B$, the corresponding X_k is a sub-disk of $K_1\backslash(U_1\cup U_2)$ and an index j can be found so that $X_j\cap D=s_j$. For this j the set $(D\backslash D_j)\cup X_j$ is a disk, and a neighborhood V of X_j can be chosen so that $V\cap(A\cup C\cup B)=\square$ and $V\cap D\cap K_1\subset X_j$. Then $(D\backslash D_j)\cup X_j$ can be deformed away from X_j semi-linearly so that no point outside V is moved, and the resulting disk D^* has the same boundary and intersection with $A\cup C\cup B$ as D but has at least one less component of intersection with K_1 which fails to link $A\cup c\cup B$. This process can be repeated and after doing so a finite number of times a new disk D is found such that each s_k in $D\cap K_1$ links $A\cup c\cup B$.

That $D\cap K_1\neq\square$ follows from the fact that the new D, like the old, contains $U\subset \text{Int } K_1$ and $\partial D\subset \text{Ext } K_1$, so there is an index i such that $D_i\cap K_1 =s_i$. Let $K_2=Y_i\cup D_i$. Since s_i links $A\cup c\cup B$, $Y_i\cap C=U_1$ and $D_i\cap C=U$, so $K_2\cap C=U_1\cup U$ and K_2 is the required set if it can be shown to be in $\mathfrak{P}^*(T, \epsilon)$. To show this it must first be shown that $\mathfrak{K}_2\in P(T, \epsilon)$, i.e., that K_2, T, and ϵ satisfy the five conditions of Definition 2.21.

1. Since $D_i\cap Y_i\subset D_i\cap K_1=s_i$ and $\partial D_i=\partial Y_i=s_i$, K_2 is a topological 2-sphere.

2. In order to prove $T\subset\text{Int } K_2$ the fact that $C\backslash(U_1\cup U)$ has at most three components is needed. To establish this the following more general statement, which will be useful later, is to be proved.

6.11. *If V_1 and V_2 are disjoint elements of $\mathfrak{T}$, then $C\backslash(V_1\cup V_2)$ has at most three components.*

For if $V_1\in\mathfrak{T}$, then $V_1=g(v\times E^{k-1})$ for some $g\in\mathfrak{H}$ and $0\leqq v\leqq 1$. Letting $I_0=\{(x, y, z)\in E^k \mid 0\leqq x<v\}$ and $I_1=\{(x, y, z)\in E^k \mid v<x\leqq 1\}$, it is seen that $C\backslash V_1$ is the union of $M_0=g(I_0\times E^{k-1})$ and $M_1=g(I_1\times E^{k-1})$. Further, $M_0\cup V_1$ and $M_1\cup V_1$ are both k-cells if v is neither 0 nor 1, and since the adjustment needed in the following arguments when $v=0$ or $v=1$ are easily supplied, the assumption $0<v<1$ is made. Since M_0 and M_1 are topologically the same it is also assumed without loss of generality that $V_2\subset M_1$. Then $C\backslash(V_1\cup V_2) =(C\backslash V_1)\backslash V_2=M_0\cup(M_1\backslash V_2)$ and if this set has more than three components $M_1\backslash V_2$ must have more than two, i.e., $M_1\backslash V_2=N_1\cup N_2\cup N_3$ where M_0, N_1, N_2, and N_3 are pairwise disjoint non-null sets and each N_i is a union of components of $M_1 \setminus V_2$. But since V_2 is in $\mathfrak{T}$, $C \setminus V_2$ has at most two components so at least two of the sets N_i, say N_1 and N_2, have closures that meet $M_0 \cup V_1$. Since Cl $N_i \subset$ Cl $M_1 = M_1 \cup V_1$, this requires that neither $V_1 \cap$ Cl N_1 nor $V_1 \cap$ Cl N_2 be null. But $M_1 \cup V_1$ is a k-cell and $V_1 \cap V_2 = \square$, so there is a set M_1' which is topologically the product of E^{k-1} and an open interval which contains all points of M_1 in some neighborhood of V_1. Thus N_1 and N_2 both meet a connected subset of $M_1 \setminus V_2$, contradicting the choice of N_1 and N_2. This proves 6.11.

Applying 6.11 to the present situation yields that $C \setminus (U_1 \cup U)$ has components $M \cup N \cup R$ and that the two components of $C \setminus U$ are $\mathcal{A} = M \cup U_1 \cup N$ and $\mathcal{B} = R$, where, it will be recalled, $\mathcal{A}$ is the component of $C \setminus U$ containing U_1 and T while $U_2 \subset \mathcal{B}$. Since $K_1 \cap C = U_1 \cup U_2$ and K_1 separates $T \subset$ Int K_1 from $A \cap C = a$ in R^3, it follows that the assumption that $M \cup A \subset$ Ext K_1 and $T \subset N \subset$ Int K_1 is but a notational convenience. Since N is connected and does not meet K_2, if $T \subset$ Ext K_2 then $N \subset$ Ext K_2. Since $M \cup A$ is an unbounded subset of the complement of K_2, $M \cup A \subset$ Ext K_2 also. But Ext K_2 is locally connected and since M and N have common limit points (in U_1), there are arbitrarily small connected sets in the complement of K_2 meeting both M and N. If these sets are sufficiently small they must lie in a neighborhood of U_1 containing no point of D and hence do not meet $K_1 \setminus K_2 \subset D$. This requires that they not meet K_1 and affords a contradiction since $M \subset$ Ext K_1 and $N \subset$ Int K_1.

3. Since both K_1 and D are locally polyhedral mod C, $K_2 \subset K_1 \cup D$ must be also.

4. By construction $K_2 \cap C = U_1 \cup U$ and both U_1 and U are in $\mathfrak{T}$.

5. If p is any point of Ext K_1, then there is a topological ray J from p in Ext K_1 and $J \cap X_i \subset J \cap K_1 = \square$. But the disk D_i is in $K_1 \cup$ Int K_1 so $J \cap D_i = \square$ also. Hence $J \cap K_2 = J \cap (X_i \cup D_i) = \square$ so J is an unbounded subset of the complement of K_2 which requires that J, and a fortiori p, be in Ext K_2. This proves Ext $K_1 \subset$ Ext K_2 so that by complementation in R^3, $K_2 \cup$ Int $K_2 \subset K_1 \cup$ Int $K_1 \subset S(T, \epsilon)$.

Thus K_2 is in $\mathfrak{P}(T, \epsilon)$ and by the proof of 5, K_2 satisfies condition 1 of Theorem 4.1 also. That the second condition of that theorem is satisfied follows immediately from the hypothesis $U_1 < T < U$. Thus $K_2 \in \mathfrak{P}^*(T, \epsilon)$.

This completes the proof of the lemma for the case $U_1 < T < U < U_2$, and the only other possible case, $U_1 < U < T < U_2$, can be made to depend on the first case in the following way. Consider the effect upon the order relation and the hypotheses and conclusion of the lemma if the names A and B are interchanged and $h^* = hr$ is used instead of h, where r is the homeomorphism of E^k onto itself defined by $r(x_1, x_2, \cdots, x_k) = (1 - x_1, x_2, \cdots, x_k)$. Evidently the new A and B are as required since T_0 and T_1 are interchanged, and since the order relation is reversed as well as the sets U_1 and U_2, the second case is reduced to the first.

6.2. Definition. *For every* $T\in\mathfrak{T}_h$ *and* $\epsilon>0$ *let* $\mathfrak{Q}(T, \epsilon)$ *denote the collection of all* $K\subset R^3$ *such that*

1. K *is a topological* 2-*sphere.*
2. $T\subset \text{Int } K$.
3. K *is locally polyhedral modulo* C.
4. $K\cap[A\cup C\cup B]=L\cup R$ *where each of the sets* L *and* R *is either an element of* $\mathfrak{T}$ *or a point of* $(A\cup B)\backslash C$.
5. $L<T<R$.
6. $K\subset S(C, \epsilon)$.

It should be noted that $\mathfrak{Q}(T, \epsilon)$ depends on h, A, and B. This dependence is not indicated in the notation since in the applications h, A, and B will be chosen and fixed.

6.3. Lemma. *Let* $0\leqq x_1<x_2\leqq 1$, $T_{x_i}=h(x_i\times E^{k-1})$, $K_i\in\mathfrak{Q}(T_{x_i}, \epsilon)$ *and* $K_i\cap(A\cup C\cup B)=L_i\cup R_i$, $i=1, 2$, *be such that* $L_1<L_2<R_1<R_2$. *Then there is a* K_3 *in* $\mathfrak{Q}(T_{x_1}, \epsilon)$ *with* $K_3\cap(A\cup C\cup B)=L_1\cup R_2$ *and* $(A\cup C\cup B)\cap\text{Int } K_3 =(A\cup C\cup B)\cap[(\text{Int } K_1)\cup(\text{Int } K_2)]$.

Proof. As before the adjustments needed for the cases where some or all of the sets L_1, R_1, L_2, R_2 are points of $(A\cup B)\backslash C$ are easily made, so only the case where all are elements of $\mathfrak{T}$ will be considered. As a preliminary step the following statement is to be proved.

6.31. If $K\in\mathfrak{Q}(T, \epsilon)$ then $(A\cup C\cup B)\cap\text{Int } K=\mathcal{B}(L)\cup\mathcal{A}(R)$.

Since $L<T<R$, L separates $C\cap A=a$ and T on C, R separates T and $b=B\cap C$, and each separates a and b on C, so the three components of $C\backslash(L\cup R)$ as guaranteed by 6.11 must be C_A, C_B, and C_T; that is, the one containing a, the one containing b, and the one containing T respectively. Thus $A\cup C\cup B=A\cup C_A\cup L\cup C_T\cup R\cup C_B\cup B$ and since $A\cup C_A$ and $C_B\cup B$ are unbounded connected sets in the complement of K, they lie in Ext K. Hence $(A\cup C\cup B)\cap\text{Int } K\subset C_T$ and, since C_T is connected, does not meet K, and contains $T\subset\text{Int } K$, the reverse inclusion also holds so

$$C_T = (A \cup C \cup B) \cap \text{Int } K.$$

Now the two components $\mathcal{A}(L)$ and $\mathcal{B}(L)$ of $(A\cup C\cup B)\backslash L$ are then $A\cup C_A$ and $C_T\cup R\cup C_B\cup B$, and since $L<T$ implies $T\subset\mathcal{B}(L)$ it follows that $\mathcal{A}(L)=A\cup C_A$ and $\mathcal{B}(L)=C_T\cup R\cup C_B\cup B$. Similarly from $T<R$ it is seen that $\mathcal{A}(R)=A\cup C_A\cup L\cup C_T$ while $\mathcal{B}(R)=C_B\cup B$. Thus $\mathcal{A}(R)\cap\mathcal{B}(L) =(A\cup C_A\cup L\cup C_T)\cap(C_T\cup R\cup C_B\cup B)=C_T$ which proves 6.31.

To proceed with the proof of Lemma 6.3, it is noted that since $L_1<L_2<R_1$ then L_2 is in both $\mathcal{B}(L_1)$ and $\mathcal{A}(R_1)$ so, by 6.31, L_2 is in Int K_1 and $K_2\cap\text{Int } K_1 \neq\square$. But $R_1<R_2$ so $R_2\subset\mathcal{B}(R_1)$ and, by 6.31, $\mathcal{B}(R_1)$ is in Ext K_1. Thus K_2 meets Ext K_2 as well as Int K_1 and $K_2\cap K_1\neq\square$ follows. Since $K_2\cap K_1\cap C =(L_2\cup R_2)\cap(L_1\cup R_1)=\square$ there is a neighborhood V of $K_1\cap K_2$ containing no points of $A\cup C\cup B$ and both K_1 and K_2 are locally polyhedral at each point of V. By shifting the vertices of K_2 lying in V a distance so small that

no point outside V is moved, K_1 and K_2 may be brought into relative general position so that $K_1 \cap K_2$ is a finite collection of mutually disjoint simple closed curves $s_1, s_2, \cdots, s_n$. Each s_j bounds a pair of sub-disks X_{ij} and Y_{ij} of K_i where X_{ij} contains L_i, $i=1, 2$.

Let an arc c be chosen in C so that it has only its end points $a=A\cap C$ and $b=C\cap B$ in common with ∂C and meets each of the sets L_1, L_2, R_1, and R_2 in a single point. Now suppose s_k links $A\cup c\cup B$. Then none of the four disks X_{1k}, X_{2k}, Y_{1k}, Y_{2k} bounded by s_k can lie in the complement of $A\cup c\cup B$ so $X_{ik}\supset L_i$ and $Y_{ik}\supset R_i$, $i=1, 2$. If on the other hand s_k does not link $A\cup c\cup B$ then each of the sets $X_{ik}\cap(A\cup c\cup B)$ and $Y_{ik}\cap(A\cup c\cup B)$ must consist of two points or no points for each choice of i, and since $L_i\subset X_{ik}$ it follows that $R_i\subset X_{ik}$ also and $Y_{ik}\cap(A\cup c\cup B)=\square$, $i=1, 2$. No other possibilities exist, since s_k either does or does not link $A\cup c\cup B$, so for any k either $R_i\subset Y_{ik}$ for both $i=1$ and $i=2$, or Y_{1k} and Y_{2k} are both disjoint from $A\cup C\cup B$.

If there is any index k such that Y_{1k} is disjoint from $A\cup C\cup B$, then there is evidently one such index j such that $Y_{1j}\cap K_2=s_j$. Then $Z=Y_{1j}\cup Y_{2j}$ is a topological 2-sphere for $Y_{1j}\cap Y_{2j}=Y_{1j}\cap K_2=s_j$ and $s_j=\partial Y_{1j}=\partial Y_{2j}$. Further, Z is polyhedral since both Y_{1j} and Y_{2j} are subsets of spheres which are locally polyhedral mod C and neither Y_{1j} nor Y_{2j} meets $A\cup C\cup B$. Beispeil III, §3, [5] can be invoked here to produce a semi-linear map of R^3 onto itself throwing Y_{1j} onto Y_{2j} and leaving $A\cup C\cup B$ fixed, but the question of how the interiors of K_1, K_2, and the interiors of their images under this map are related would then arise. To avoid this difficulty an isotopy on R^3 achieving the above result is to be constructed.

There is a semi-linear homeomorphism f of R^3 onto itself throwing Z onto the surface of the unit cube E in R^3 and evidently it may be supposed that if E^+ and E^- denote the portions of the surface of E on and above and on and below the xy-plane respectively, then $f(Y_{1j})=E^+$ and $f(Y_{2j})=E^-$.

An isotopy g_t $(0\leqq t\leqq 1)$ of R^3 onto itself which throws E^- onto E^+ and moves no point outside $S(E, \delta)$ with δ arbitrarily small can be defined with no difficulty, and there is no loss in taking g_1 semilinear. Then any point of $f(A\cup C\cup B)$ which is interior (exterior) to $f(K_2)$ at the stage $t=0$ remains interior (exterior) to $g_tf(K_2)$ for each t, so that $f^{-1}g_1f(K_2)$ has all the properties required of the original K_2. The δ may be taken so small that no point of $f(K_2)$ lies interior to $S(E, \delta)$ except $f(Y_{2j})$ and an arbitrarily small neighborhood of $f(Y_{2j})$ relative to $f(K_2)$, and the isotopy may be defined so as to move no point of $f(K_2)$ except $f(Y_{2j})$.

Consequently $f^{-1}g_1f(K_2)$ is $(K_2\setminus Y_{2j})\cup Y_{1j}$ and this sphere may be deformed semi-linearly away from K_1 in a neighborhood of Y_{1j} so small that no point of $A\cup C\cup B$ is moved. The resulting sphere is a new K_2 which has all the properties required of the old and which has at least one less intersection s_k with K_1 which does not link $A\cup c\cup B$.

Thus after a finite number of repetitions a new K_2 is obtained such that every component of its intersection with K_1 links $A\cup c\cup B$. Then each remaining X_{1k} contains L_1 and not R_1, and an index p is found such that $X_{1p}\cap K_2=s_p$. Let $K_3=X_{1p}\cup Y_{2p}$, and note that $K_3\cap(A\cup C\cup B)=[X_{1p}\cap(A\cup C\cup B)]\cup[Y_{2p}\cap(A\cup C\cup B)]=L_1\cup R_2$. Thus in view of 6.31, K_3 is the desired sphere provided it can be shown to satisfy the six conditions of Definition 6.2 for being an element of $\mathfrak{Q}(T_{x_1}, \epsilon)$.

1. Since $X_{1p}\cap Y_{2p}=s_p=\partial X_{1p}=\partial Y_{2p}$, K_3 is a topological 2-sphere.

2. The proof that $T_{x_1}\subset\text{Int } K_3$ is essentially a repetition of the argument given to show $T\subset\text{Int } K_1$ in the proof of Lemma 6.1 and hence need only be outlined here. From $K_1\in\mathfrak{Q}(T_{x_1}, \epsilon)$ and $L_1<L_2<R_1<R_2$ it is seen that $\mathcal{A}(T_{x_1})\backslash L_1$ has two components, $\mathcal{A}(L_1)$ in Ext K_3 and N containing T_{x_1}. If $T_{x_1}\subset\text{Ext } K_3$, then $N\subset\text{Ext } K_3$ and connected subsets of Ext K_3 meeting both N and $\mathcal{A}(L_1)$ with arbitrarily small diameters exist, for $(\text{Cl } N)\cap\text{Cl }\mathcal{A}(L_1)=L_1\neq\square$. But if the diameters of such sets are sufficiently small they are seen to lie in a neighborhood of L_1 in which K_3 and K_1 are identical, and hence lie in the complement of K_1. This contradicts $T_{x_1}\subset\text{Int } K_1$, which cannot be false for $K_1\in\mathfrak{Q}(T_{x_1}, \epsilon)$.

3. $K_3\subset K_1\cup K_2$ and hence is locally polyhedral modulo C for both K_1 and K_2 are.

4. $K_3\cap(A\cup C\cup B)=L_1\cup R_2$, and these are as required since they are components of intersection of $A\cup C\cup B$ and K_1 and K_2 respectively.

5. That $L_1<T_{x_1}<R_2$ holds was shown above.

6. $K_3\subset K_1\cup K_2\subset S(C, \epsilon)$ as required.

This completes the proof.

7. The enclosure property.

7.1. THEOREM. *If there is an $h\in\mathfrak{H}$ such that C has property $\mathfrak{P}$ relative to $\mathfrak{T}_h$ and the uniform disk property relative to $\mathfrak{T}_h$, then C has the enclosure property.*

Proof. Let $\epsilon>0$ be assigned. Let A' and B' be a pair of topological rays with initial points $a=A'\cap C$ in $T_0\backslash\partial T_0$ and $b=B'\cap C$ in $T_1\backslash\partial T_1$ respectively. That such a pair of rays which are mutually disjoint and locally polyhedral mod C exist was shown in the preceeding section. Let ω_1 denote min $[\epsilon, d(T_0, T_1)/2, d(T_0, B'), d(T_1, A')]$ and choose K_0 in $\mathfrak{P}^*(T_0, \omega_1)$ and K_1 in $\mathfrak{P}^*(T_1, \omega_1)$. Let A'' be the sub-arc of A' which is minimal with respect to containing $K_0\cap A'$. By the choice of ω_1 and the fact that $T_0\subset\partial C$, $K_0\cap C=R_0\in\mathfrak{T}$ and $K_0\backslash C$ is an open disk so there is an arc A''' in $K_0\backslash C$ with the same end points as A''. The arc A''' may be taken polyhedral and by further subdividing it and shifting all of its vertices except one end point into Ext K_0 a sufficiently small distance so as to introduce no new points of intersection with $B\cup C\cup K_1$ the ray $(A'\backslash A'')\cup A'''$ is deformed into a ray A such that $A\cap K_1=\square$, $A\cap K_0=L_0$, a point, and $A\cap C=a$, the initial point of A. A ray B is constructed from B' in a similar manner so that $B\cap K_0=\square$, $B\cap K_1=R_1$, a point, and $B\cap C=b$, the initial point of B.

Now K_0 and K_1 are seen to be in $\mathfrak{Q}(T_0, \omega_1)$ and $\mathfrak{Q}(T_1, \omega_1)$ respectively, for all of the six conditions of 6.2 are satisfied by virtue of the fact that K_i is in $\mathfrak{P}^*(T_i, \omega_1)$, $i=0, 1$, except condition 4: $K_i \cap (A \cup C \cup B) = L_i \cup R_i$ where each of the sets L_i and R_i is either an element of $\mathfrak{T}$ or a point of $(A \cup B) \setminus C$. But $K_0 \cap (A \cup C \cup B) = (K_0 \cap A) \cup (K_0 \cap C) \cup (K_0 \cap B) = L_0 \cup R_0 \cup \square$ where L_0 is a point of A and R_0 is in $\mathfrak{T}$ by construction. Similarly $K_1 \cap (A \cup C \cup B) = \square \cup L_1 \cup R_1$ where L_1 is in $\mathfrak{T}$ and R_1 is a point of B.

Since $T_0 \subset (\text{Int } K_0) \cap (\text{Ext } K_1)$ and $T_1 \subset (\text{Int } K_1) \cap (\text{Ext } K_0)$ it is seen that two real numbers p, q with $0 < p < 1/2 < q < 1$ can be chosen so that $T_x \subset \text{Int } K_0$ for $0 \leqq x \leqq p$ and $T_x \subset \text{Int } K_1$ for $q \leqq x \leqq 1$. Let such a p and q be chosen and let M denote $\bigcup_{p \leqq x \leqq q} T_x$. The plan of proof will be first to show there is a K_p in $\mathfrak{Q}(T_p, \epsilon)$ such that $M \subset \text{Int } K_p$, and then to apply Lemma 6.3 twice, first to K_0 and K_p and then to the resulting sphere and K_1, to obtain the desired sphere. In order to construct K_p some preliminary steps are necessary.

If $\mathfrak{T}_m$ denotes the subset of $\mathfrak{T}_h$ consisting of all T_x in $\mathfrak{T}_h$ with $p \leqq x \leqq q$, $\mathfrak{T}_m$ is evidently a sub-arc of $\mathfrak{T}_h$ and hence is compact. If we let ω_2 denote Min $[\omega_1, d(M, T_0 \cup T_1 \cup A \cup B)]$ and $\omega_3 > 0$ be the number corresponding to ω_2 guaranteed by the hypothesis that C has the uniform disk property relative to $\mathfrak{T}_h$; that is let ω_3 be such that if $T_z \in \mathfrak{T}_h$ and $\eta > 0$ there is a D in $\mathfrak{D}(T_z, \eta)$ with $d(\partial D, C) > \omega_3$ and $D \subset S(T_z, \omega_2)$. Now, since $\mathfrak{T}_m$ is compact and C has property $\mathcal{P}$ relative to $\mathfrak{T}_h$ (and hence relative to $\mathfrak{T}_m$), there is a $\gamma > 0$ such that for every $T_x \in \mathfrak{T}_m$, i.e., every $p \leqq x \leqq q$, $\sup_{K \in \mathfrak{P}^*(T_x, \omega_3)} d(T_x, K) > 2\gamma$. Evidently then there is a K_x in $\mathfrak{P}^*(T_x, \omega_3)$ such that $T_y \subset \text{Int } K_x$ for all y such that $\rho(T_x, T_y) < \gamma$. Since h is a homeomorphism, it is easily seen that there is an $\alpha > 0$ corresponding to γ such that whenever $|x-y| \leqq \alpha$ then $\rho(T_x, T_y) < \gamma$. Without loss of generality it may be supposed that $p - \alpha > 0$ and $q + \alpha < 1$.

Let $x_1 = p - \alpha$, $x_2 = p$, $x_3 = p + \alpha, \cdots$, $x_i = p + (i-2)\alpha, \cdots$, $x_{j+1} = p + (j-1)\alpha$, where j is chosen so that $x_j < q \leqq x_{j+1}$. For $i = 2, 3, \cdots, j$, choose $\mathfrak{K}_i'$ in $P^*(T_{x_i}, \omega_3)$ such that $T_y \subset \text{Int } K_i$ for all y with $x_{i-1} \leqq y \leqq x_{i+1}$. This is possible for if $x_{i-1} \leqq y \leqq x_{i+1}$ then $|x_i - y| \leqq \alpha$ and $\rho(T_{x_i}, T_y) < \gamma$.

It will now be shown that from K_i' a sphere K_i in $\mathfrak{P}^*(T_{x_i}, \omega_3)$ can be formed with the property that $K_i \cap C = L_i \cup R_i$ with $\rho(L_i, T_{x_{i-1}})$ and $\rho(R_i, T_{x_{i+1}})$ both arbitrarily small, $i = 2, 3, \cdots, j$. To see this let $\beta > 0$ be assigned. For each $i = 1, \cdots, j+1$ choose $D_i \in D(T_{x_i}, \beta)$ such that $d(\partial D_i, C) > \omega_3$ and $D_i \subset S(T_{x_i}, \omega_2)$. Then D_{i+1}, K_i', T_{x_i}, and ω_3 satisfy the seven hypotheses of Lemma 6.1, as follows.

1. T_{x_i} is in $\mathfrak{T}$ and separates $A \cap C$ and $B \cap C$ on $A \cup C \cup B$ since each element of $\mathfrak{T}_h$ has these properties.

2. $K_i' \in \mathfrak{P}^*(T_{x_i}, \omega_3)$ by choice.

3. Since $(K_i' \cup \text{Int } K_i') \subset S(T_{x_i}, \omega_3)$ and $\omega_3 < d(T_{x_i}, A \cup B)$, both A and B are separated from T_{x_i} by $K_i' \cap C$ on $A \cup C \cup B$. Hence $C \cap K_i'$ has two components and consists of a pair of elements L_i' and R_i' of $\mathfrak{T}$.

4. From the above, if the notation is properly chosen, L_i' separates A

and T_{x_i} while R_i' separates T_{x_i} and B on $A\cup C\cup B$. By 2 of Theorem 4.1 and Definition 4.2, T_{x_i} separates L_i' and R_i' on $A\cup C\cup B$, so $L_i' < T_{x_i} < R_i'$. Since $T_y \subset \text{Int } K_i'$ for all y with $|x_i - y| \leqq \alpha$ and since $x_{i-1}+\alpha = x_i = x_{i+1}-\alpha$, $L_i' < T_{x_{i-1}} < T_{x_i} < T_{x_{i+1}} < R_i'$. Letting U_i denote $D_i \cap C$, $i=0, 1, \cdots, j+1$, and noting that $\rho(U_i, T_{x_i}) < \beta$, it is seen from Lemma 3.1 that if a sufficiently small number is used in place of β then $L_i' < U_{i-1} < T_{x_i} < U_{i+1} < R_i'$.

5. Since $U_{i+1} = D_{i+1} \cap C$, then $\rho(U_{i+1}, C\cap D_{i+1}) < \omega$ for every $\omega > 0$ so D_{i+1} satisfies condition 5 of Definition 2.31 for being in $\mathfrak{D}(U_{i+1}, \omega)$ for every $\omega > 0$. That D_{i+1} also satisfies all the other conditions of 2.31 follows from the fact that D_{i+1} was chosen in $\mathfrak{D}(T_{x_{i+1}}, \beta)$.

6. Since $D_{i+1} \subset S(T_{x_{i+1}}, \omega_2)$ and $K_i \in \mathfrak{P}^*(T_{x_i}, \omega_3)$, then $D_{i+1}\cup K_i \subset S(M, \omega_2)$ for $D_{i+1}\cap C$, $T_{x_{i+1}}$, and T_{x_i} are all in M and $\omega_2 > \omega_3$. But $\omega_2 < d(M, A\cup B)$ so $(D_{i+1}\cup K_i)\cap(A\cup B) = \square$.

7. By choice of D_{i+1}, $d(\partial D_{i+1}, C) > \omega_3$.

Thus Lemma 6.1 applies so there is a K_i'' in $\mathfrak{P}^*(T_{x_i}, \omega_3)$ with $K_i'' \cap C = L_i' \cup U_{i+1}$. In the same way it is verified that D_{i-1}, K_i'', T_{x_i}, and ω_3 also satisfy the hypothesis of Lemma 6.1 with $L_i' < U_{i-1} < T_{x_i} < U_{i+1}$, so there is a K_i in $\mathfrak{P}^*(T_{x_i}, \omega_3)$ with $K_i \cap C = U_{i-1} \cup U_{i+1}$, $i=2, 3, \cdots, j$.

Next it must be shown that K_i is in $\mathfrak{Q}(T_{x_i}, \omega_3)$, $i=2, \cdots, j$. The first three conditions of Definition 6.2 are satisfied by K_i, T_{x_i}, and ω_3 by virtue of $K_i \in \mathfrak{P}^*(T_{x_i}, \omega_3)$, while conditions 4 and 5 follow from the construction. Condition 6 also holds, since $S(C, \omega_3) \supset S(T_{x_i}, \omega_3) \supset K_i$.

Now Lemma 6.3 is to be applied to K_2 and K_3. Since $0 < x_2 < x_3 = x_2 + \alpha < 1$, and since each K_i has been shown to be in $\mathfrak{Q}(T_{x_i}, \omega_3)$, Lemma 6.3 applies provided $U_1 < U_2 < U_3 < U_4$. But $U_i = D_i \cap C$ and $\rho(T_{x_i}, U_i) < \beta$, so if β was taken sufficiently small then $U_1 < U_2 < \cdots < U_{j+1}$ follows from $T_{x_1} < T_{x_2} < \cdots < T_{x_{j+1}}$ and Lemma 3.1. Thus there is a K_{23} in $\mathfrak{Q}(T_{x_1}, \omega_3)$ with $K_{23}\cap C = U_1 \cup U_4$ and $(A\cup C\cup B)\cap \text{Int } K_{23} = [A\cup C\cup B]\cap[(\text{Int } K_2)\cup(\text{Int } K_3)]$. This process is repeated with K_{23} and K_4 to obtain K_{24}, with K_{24} and K_5 to obtain K_{25}, and so on until a sphere K_{2j} in $\mathfrak{Q}(T_{x_1}, \omega_3)$ is obtained with $K_{2j}\cap C = U_1 \cup U_{j+1}$ and $(A\cup C\cup B)\cap \text{Int } K_{2j} = (A\cup C\cup B)\cap[(\text{Int } K_1)\cup \cdots \cup(\text{Int } K_j)]$. Every point of M lies in a T_x for some x with $p \leqq x \leqq q$ and hence with $x_1 \leqq x \leqq x_j$ so $M \subset \text{Int } K_{2j}$.

Thus three spheres have been obtained, first K_0 in $\mathfrak{Q}(T_0, \omega_1)$ with $K_0 \cap (A\cup C\cup B) = L_0 \cup R_0$ where L_0 is a point of A and R_0 separates T_p and T_q on $A\cup C\cup B$, second K_{2j} in $\mathfrak{Q}(T_{x_1}, \omega_3)$ with $K_{2j}\cap(A\cup C\cup B) = U_1 \cup U_{j+1}$ where U_1 separates T_0 and T_p $(=T_{x_2})$ on $(A\cup C\cup B)$ and U_{j+1} separates T_q (for $q \leqq x_{j+1}$) and T_1 on $(A\cup C\cup B)$, and third K_1 in $\mathfrak{Q}(T_1, \omega_1)$ with $K_1 \cap (A\cup C\cup B) = L_1 \cup R_1$ where L_1 separates T_p and T_q on $(A\cup C\cup B)$ and R_1 is a point of B. Hence $L_0 < U_1 < R_0 < U_{j+1}$ and $L_0 < L_1 < U_{j+1} < R_1$ so, since $0 < p < 1$, Lemma 6.3 can be applied, first to K_0 and K_{2j} to obtain a K' in $\mathfrak{Q}(T_0, \omega_1)$ with $K' \cap (A\cup C\cup B) = L_0 \cup U_{j+1}$, and then to K' and K_1 to obtain a K in $\mathfrak{Q}(T_0, \omega_1)$ with $K\cap(A\cup C\cup B) = L_0 \cup R_1$.

Then $(A\cup C\cup B)\cap\text{Int } K$ is the union of those portions of $(A\cup C\cup B)$ which were in the interior of any one of the three spheres K_0, K_{2j}, and K_1. But any point of C lies in T_x for some x, and hence is interior to K_0, K_{2j}, or K_1 according as $0\leqq x\leqq p$, $p\leqq x\leqq q$, or $q\leqq x\leqq 1$. Thus $C\subset\text{Int } K$. Also, since $K\in\mathfrak{Q}(T_0, \omega_1)$, $K\subset S(C, \omega_1)\subset S(C, \epsilon)$. That K is polyhedral follows from the fact that it is locally polyhedral modulo C and does not meet C. This completes the proof, but a fact that will prove useful later should be noted here; i.e., the sphere K meets each of the rays A, B in a single point.

8. **The strong enclosure property.** For the standard cell E^k and a given $\epsilon>0$ it is evident that not only is there a polyhedral 2-sphere M in $S(E^k, \epsilon)$ with $E^k\subset\text{Int } M$, but the sphere M may be taken to be the boundary of a convex 3-cell so that a straight ray meeting E^k only at its initial point meets M in a single point. Consequently any tame cell will have the following property.

8.1. DEFINITION. *A cell C is said to have the strong enclosure property provided to each $h\in\mathfrak{H}$ there corresponds a pair of disjoint topological rays A, B and a sequence of polyhedral 2-spheres $\{M_i\}$ which meet the following conditions:*

1. *$A(B)$ meets C only at its initial point $a\in T_0\backslash\partial T_0(b\in T_1\backslash\partial T_1)$.*
2. *$A\cup B$ is locally polyhedral modulo C.*
3. *$M_i\subset S(C, 1/i)$ and $C\subset\text{Int } M_i$, $i=1, 2, \cdots$.*
4. *M_i meets each of the rays A, B in a single point, $i=1, 2, \cdots$.*

Whether or not a cell with the enclosure property can fail to have the strong enclosure property is an unanswered question, although it has already been shown that the sets A, B, and $\{M_i\}$ can be chosen satisfying all conditions except possibly condition 4 whenever C has the enclosure property.

8.2. THEOREM. *If C is a cell with property $\mathcal{P}$, then a necessary and sufficient condition that C have the strong enclosure property is that C have the uniform disk property.*

Proof of sufficiency. Let $h\in\mathfrak{H}$ be assigned and choose rays A_0, B_0 satisfying all the conditions of Definition 8.1 except possibly condition 4. Let ϵ_i be a null sequence of positive numbers, and for each integer i let T_{0i} be a $(k-1)$-cell such that (1) $T_{0i}\backslash\partial T_{0i}$ contains $a=A_0\cap C$, (2) $T_{0i}\subset T_0$, and (3) $T_{0i}\subset S(a, \epsilon_i)$. Also for each i let T_{1i} be a $(k-1)$-cell with similar relationships with $b=B_0\cap C$ and T_1. Then for each i a homeomorphism $h_i\in\mathfrak{H}$ can be chosen so that $h_i(0\times E^{k-1})=T_{0i}$ and $h_i(1\times E^{k-i})=T_{1i}$. (In case C is a 1-cell each h_i may be taken to be h itself, for then $T_{0i}=T_0$ and $T_{1i}=T_1$.)

Now for each successive i the cell C has property $\mathcal{P}$ relative to $\mathfrak{T}_{h_i}$ and the uniform disk property relative to $\mathfrak{T}_{h_i}$, so the construction of Theorem 7.1 with ϵ_i used as the ϵ can be carried out. It is recalled that in this construction the original rays (call them A_{i-1} and B_{i-1}) were replaced by a new pair (call them A_i and B_i) which had the property that $A_i\backslash A_{i-1}$ $(B_i\backslash B_{i-1})$ was an arc obtained by deforming a sub-arc of a sphere K_0 in $S(T_{0i}, \epsilon_i)$ $[K_1$ in $S(T_{1i}, \epsilon_i)]$

an arbitrarily small distance, so that $A_i \backslash A_{i-1} \subset S(T_{0i}, \epsilon_i)$ and $B_i \backslash B_{i-1} \subset S(T_{1i}, \epsilon_i)$ may be assumed. The result of this construction was a polyhedral sphere (say M_i) in $S(C, \epsilon_i)$ with $C \subset \text{Int } M_i$ which met each of the rays A_i, B_i in a single point. Evidently the sequence ϵ_i can be chosen so that $M_i \subset S(C, 1/i)$ and also sufficiently small that any sequence $\{a_i\}$ with $a_i \in A_i \backslash A_{i-1}$ must have limit a. To see the latter it is only necessary to note that if $a_i \in A_i \backslash A_{i-1}$ then $a_i \in S(T_{0i}, \epsilon_i)$ so there is an a_i^* in T_{0i} such that $d(a_i, a) \leqq d(a_i, a_i^*) + d(a_i^*, a) \leqq \epsilon_i + d(a_i^*, a)$. But $d(a_i^*, a) < \epsilon_i$, for $T_{0i} \subset S(a, \epsilon_i)$ by choice, so $d(a_i, a) \leqq 2\epsilon_i$. Similarly any sequence $\{b_i\}$ with $b_i \in B_i \backslash B_{i-1}$ may be supposed to have limit b.

At the ith stage of this iterative construction A_i (B_i) is formed by replacing a sub-arc A_i^* (B_i^*) of A_{i-1} (B_{i-1}) by an arc $A_i^{**} = A_i \backslash A_{i-1}$ $(B_i^{**} = B_i \backslash B_{i-1})$. Let $A = [A_0 \backslash \bigcup A_i^*] \cup \bigcup A_i^{**}$ and $B = [B_0 \backslash \bigcup B_i^*] \cup \bigcup B_i^{**}$. If the sequence $\{\epsilon_i\}$ is properly chosen, then A and A_i are identical on a sub-ray of A which includes all points of A that are separated from a by A_i^{**}, so that $A \backslash a$ is topologically a ray with the initial point deleted. This implies that A is a ray with initial point a provided $A = \text{Cl } (A \backslash a)$, and this equality follows from the fact that $\lim a_i = a$ for every sequence $\{a_i\}$ with $a_i \in A_i^{**}$. Similarly B is a ray.

Thus the sufficiency is established provided A, B, and the chosen sequence $\{M_i\}$ satisfy condition 4 of Definition 8.1. But $M_i \cap A$ is the same as $M_i \cap A_i$ provided A and A_i are the same exterior to a neighborhood of a which lies in Int M_i, and this can be guaranteed by choosing ϵ_{i+1} sufficiently small.

Proof of necessity. Let $h \in \mathfrak{H}$ and $\omega > 0$ be assigned. In order to show that C has the uniform disk property it is necessary to exhibit a $\delta > 0$ such that for every $T \in \mathfrak{T}_h$ and $\epsilon > 0$ there is a D in $\mathfrak{D}(T, \epsilon)$ with $d(\partial D, C) > \delta$ and $D \subset S(T, \omega)$. First the following statement must be established.

8.21. There is a $\beta > 0$ such that if U and T are any pair of elements of $\mathfrak{T}_h$ with $U \subset S(T, \beta)$, then $\rho(T, U) < \omega/2$.

If 8.21 is denied, then for each integer i there is a pair U_i, T_i of elements of $\mathfrak{T}_h$ with $U_i \subset S(T, 1/i)$ and $\rho(T_i, U_i) \geqq \omega/2$. Since $\mathfrak{T}_h$ is an arc under the metric ρ, it may be assumed that $\{T_i\}$ and $\{U_i\}$, considered as elements of $\mathfrak{T}_h$, converge to $T \in \mathfrak{T}_h$ and $U \in \mathfrak{T}_h$ respectively. Since ρ is a metric, $\rho(T, U) \geqq \omega/2$, and since $\mathfrak{T}_h$ is an arc, T and U must be distinct elements of $\mathfrak{T}_h$ and hence disjoint sets in R^3. Then $\alpha = d(T, U)$ is positive, and by definition of ρ, the inclusions $T_i \subset S(T, \alpha/4)$ and $U_i \subset S(U, \alpha/4)$ must hold for all i greater than some fixed N. But if $1/i < \alpha/4$ and $i > N$, then $U_i \subset S(T_i, \alpha/4)$ and hence $U_i \subset S(T, \alpha/2)$, contradicting $U_i \subset S(U, \alpha/4)$ since these two sets are disjoint. From 8.21 the following will be derived.

8.22. There is an $\eta > 0$ such that if U and T are any pair of elements of $\mathfrak{T}_h$ and $K \in \mathfrak{P}^*(T, \eta)$, then $U \subset \text{Int } K$ implies $K \in \mathfrak{P}^*(U, \omega)$.

It is evident that no matter what $\eta > 0$ is chosen K must satisfy all of the requirements of Definition 2.21 for being in $\mathfrak{P}(U, \omega)$ except possibly $K \subset S(U, \omega)$, and that U must separate the components of $K \cap C$ whenever there are two, since T does so and both U and T are in $\mathfrak{T}_h$. Hence 8.22 follows if η can be chosen so that $(K\cup\text{Int } K)\subset S(U,\omega)$ whenever $(K\cup\text{Int } K)\subset S(T,\eta)$. Let $\eta=\min(\omega/2,\beta)$, where β is the number required by 8.21, and suppose $p\in(K\cup\text{Int } K)\subset S(T,\eta)$. Then $d(p,U)\leqq d(p,q)+d(q,U)$ for any point q of T, so $d(p,U)\leqq\eta+\sup_{q\in T} d(q,U)\leqq\omega/2+\rho(T,U)\leqq\omega/2+\omega/2=\omega$, and 8.22 follows.

Now since C has the strong enclosure property, two rays A' and B' and a sequence $\{M_i\}$ of polyhedral spheres satisfying Definition 8.1 can be chosen. Let $\alpha=\min[\eta, d(T_0,T_1)/2, d(T_0,B'), d(A',T_1)]$ and choose K_0 in $\mathfrak{P}^*(T_0,\alpha)$ and K_p in $\mathfrak{P}^*(T_1,\alpha)$, where the index p will be explained below. The rays A' and B' are used in the now familiar way to form rays A and B such that $A\cap K_0$ is a point, $A\cap K_p=\square$, $B\cap K_0=\square$, and $B\cap K_p$ is a point. Since A and B are identical with A' and B' in some neighborhood of C, it may be supposed that A, B, and $\{M_i\}$ satisfy Definition 8.1, for deleting a finite number of the M_i would make this true.

For each $U\in\mathfrak{T}_h$ except T_0 and T_1, $\alpha(U)=\min[\eta, d(U,A\cup B)]$ is positive, and $K(U)\in\mathfrak{P}^*(U,\alpha[U])$ can be chosen. Let $M(U)$ denote the set of all $T\in\mathfrak{T}_h$ such that $T\subset\text{Int } K(U)$, and take $K_0=K(T_0)$ and $K_p=K(T_1)$. Then the collection of all $M(U)$ is an open covering of the arc $\mathfrak{T}_h$ and corresponding to the finite sub-covering which must exist is the collection $K_0, K_1, \cdots, K_p$. [K_0 and K_p must be present, since T_0 and T_1 lie in no $M(U)$ except $M(T_0)$ and $M(T_1)$ respectively, by the choice of $\alpha(U)$.] This collection and the arcs A and B are seen to have the following properties.

1. A, B, and $\{M_i\}$ satisfy Definition 8.1.
2. $A\cap K_i=\square$ or L_0, a point, according as $i>0$ or $i=0$, and $B\cap K_i=\square$ or R_p, a point, according as $i<p$ or $i=p$.
3. $K_0\cap K_p=\square$ and $K_i\cap(T_0\cup T_1)=\square$ for all i.
4. For each T in $\mathfrak{T}_h$ there is an index i such that $T\subset\text{Int } K_i$, and hence $K_i\cup\text{Int } K_i\subset S(T,\omega)$ by the choice of η.

Now for each $i=0, 1, \cdots, p$ the set $K_i\cap(A\cup C\cup B)$ is the union of two components L_i and R_i, and each L_i, R_i is in $\mathfrak{T}$ except that L_0 is a point of A and R_p is a point of B. A collection $\sigma_0, \sigma_1, \cdots, \sigma_p$ of polyhedral simple closed curves can therefore be chosen so that σ_i separates L_i and R_i on K_i, $i=0, 1, \cdots, p$. It will be shown that $\delta=d(C, \sigma_0\cup\cdots\cup\sigma_p)$ is the number corresponding to the assigned ω required for the uniform disk property relative to $\mathfrak{T}_h$.

For let $T\in\mathfrak{T}_h$ and $\epsilon>0$ be assigned. Then an index j can be chosen so that $T\subset\text{Int } K_j$. Corresponding to ϵ there is an $\epsilon_1>0$ such that if D is in $\mathfrak{D}(T,\epsilon_1)$ then D is also in $\mathfrak{D}(T,\epsilon)$ and $D\cap C\subset\text{Int } K_j$. It may be assumed that $D\subset\text{Int } K_j$, for some sub-disk of the original D is in $\mathfrak{D}(T,\epsilon)$ and has this property. From the sequence $\{M_i\}$ choose a sphere M which separates C from $\partial D\cup\sigma_1\cup\cdots\cup\sigma_p$ and meets A and B in the points a and b respectively. Then

M and D may be taken in relative general position so that $M \cap D$ (which is non-null) is the union of a finite collection of mutually disjoint simple closed curves $s_1, \cdots, s_n$. Each s_i bounds a disk D_i on D and a pair of disks X_i and Y_i on M, where X_i contains the point $a = M \cap A$. If there is an i for which $D_i \cap C = \square$, then the theory of linkages may be used as before to show that $Y_i \cap (A \cup C \cup B) = \square$, and M may be replaced by the result of deforming $(M \setminus Y_i) \cup D_i$ semi-linearly away from D so as to reduce the number of indices i for which $D_i \cap C = \square$. The fact that $(D_i \cup Y_i) \cap (A \cup C \cup B) = \square$ is used to guarantee that the new M retains the property $C \subset \text{Int } M$. After a finite number of repetitions, each remaining D_i contains $D \cap C$, so that they are simply ordered by set inclusion and D can be replaced by D_1, the minimal remaining D_i. Then $D_1 \in \mathfrak{D}(T, \epsilon)$, $D_1 \subset \text{Int } K_j$, and $D_1 \cap M = s_1 = \partial D_1$.

Now M and K_j are taken in relative general position and application of the theory of linkages yields the result that any simple closed curves on $K_j \cap M$ which bound a sub-disk of $K_j \setminus (A \cup C \cup B)$ also bound a sub-disk of $M \setminus (A \cup C \cup B)$. A new M is formed for which the number of such curves of intersection with K_j is less than before. It must be noted that ∂D_1 is a subset of the new M and that σ_j does not meet it. Repetition removes all such components of $M \cap K_j$ so that $M \cap K_j$ becomes the union of a finite collection of mutually disjoint simple closed curves $s_1', s_2', \cdots, s_m'$, each of which separates a and b on M as well as L_j and R_j on K_j. Then $\partial D_1 = s_1$ is interior to K_j and lies on M, so that $s_1 \cap s_i' = \square$ for every choice of i and the curves have a natural order $s_1', s_2', \cdots, s_q', s_1, s_{q+1}', \cdots, s_m'$ induced by set inclusion of the sub-disks of $M \setminus b$ which they bound. By the choice of this order s_q' and s_1 together separate M into three components, an open disk containing a, an open disk containing b, and an open annular ring R such that $R \cap K_j = \square$ and $\text{Cl } R = R \cup s_q' \cup s_1$. Since $s_1 \subset \text{Int } K_j$, $\text{Cl } R$ is contained in $K_j \cup \text{Int } K_j$. Similarly s_q' and σ_j bound an annular ring R^* on K_j. Evidently $D^* = D_1 \cup R \cup R^*$ is a disk which is contained in $K_j \cup \text{Int } K_j$ and hence in $S(T, \omega)$ and $\partial D^* = \sigma_j$ so that $d(C, D^*) > \delta$. Thus D^* is the required disk provided it can be shown to be in $D(T, \epsilon)$.

To show this it must be verified that D^*, T, and ϵ satisfy the six conditions of Definition 2.31. Condition 1 is fulfilled by construction and condition 2 follows from the fact that $\partial D^* = \sigma_j$ was chosen to separate L_j and R_j, the components of $K_j \cap C$, on K_j. Since $D^* \cap C = D_1 \cap C = D \cap C$, condition 3 is fulfilled. Since $D^* \subset (D \cup M \cup K_j)$ and all three of these sets are locally polyhedral modulo C, so must D^* be. This is condition 4, and condition 5 follows from $D^* \cap C = D \cap C$ and $D \in \mathfrak{D}(T, \epsilon)$. Since there is some neighborhood U of $D^* \cap C$ such that $U \cap D^* = U \cap D$, condition 6 for D^* follows also from $D \in \mathfrak{D}(T, \epsilon)$. This completes the proof.

8.3. THEOREM. *If C has the strong enclosure property and the disk property, then C has property $\mathfrak{P}$.*

Proof. Let $T\in\mathfrak{T}$ and $\epsilon>0$ be assigned. Only the case $T\cap\partial C=\partial T$ will be considered since the adjustments to the following argument needed when $T\cap\partial C=T$ are easily made. Therefore there is an $h\in\mathfrak{H}$ and a number p with $0<p<1$ such that $T=h(p\times E^{k-1})$. For any pair of numbers m, n with $0<m<p<n<1$, a triple of 1-cells $I_0=\{x\in E^1|0\leqq x\leqq m\}$, $I_1=\{x\in E^1|n\leqq x\leqq 1\}$, and $I_2=\{x\in E^1|m\leqq x\leqq n\}$ is defined. This in turn determines a triple of sub-cells of C of dimension k, $C_i=h(I_i\times E^{k-1})$, $i=0$, 1, and 2. Evidently $T_0\subset C_0$, $T_1\subset C_1$, and $T\subset C_2$, and if $p-m$ and $n-p$ are taken sufficiently small, $C_2\subset S(T,\epsilon/2)$. Now choose another pair of numbers r, s so that $m<r<p<s<n$, and note that T_r separates C_0 and T on C while T_s separates T and C_1 on C.

Since C has the disk property a pair of disks $D_r\in\mathfrak{D}(T_r,\beta)$ and $D_s\in\mathfrak{D}(T_s,\beta)$ may be chosen, and by Lemma 3.1 if β is taken sufficiently small then D_r (D_s) separates T and T_m (T_n) on C. This requires that D_r (D_s) separate T and C_0 (C_1) on C. Also $R=D_r\cap C$ and $S=D_s\cap C$ are a pair of elements of $\mathfrak{T}$ which lie in C_2 and hence it may be supposed that $D_r\cup D_s$ is in $S(T,\epsilon)$, since this can be assured by taking sub-disks of the original ones.

A pair of rays A, B, and a sequence of spheres $\{M_i\}$ satisfying Definition 8.1 are now chosen, and again taking sub-disks of D_r and D_s as new disks D_r, D_s the relation $(D_r\cup D_s)\cap(A\cup B)=\square$ can be made to hold. Now let δ be min $[\epsilon/2, d(C,\partial D_r\cup\partial D_s)]$ and choose a polyhedral 2-sphere M from the sequence $\{M_i\}$ so that M is in $S(C,\delta)$, $C\subset\operatorname{Int} M$, and M meets A and B in a single point each. Taking M and $D_r\cup D_s$ in relative general position, using the theory of linkages, and replacing sub-disks of $M\backslash(A\cup B)$ by sub-disks of $(D_r\cup D_s)\backslash C$, a new M is formed such that each component of $M\cap(D_r\cup D_s)$ is a simple closed curve separating $a=M\cap A$ and $b=M\cap B$ on M. Since R separates a and S on C, if the δ above was chosen sufficiently small then D_r separates a and D_s on $S(C,\delta)$ and hence on M. But this requires that some component s_i of $D_r\cap M$ separate a and $D_s\cap M$ on M and hence a simple closed curve s_j of $D_r\cap M$ can be found such that if X_j is the sub-disk of $M\backslash D_s$ bounded by s_j then $X_j\cap D_r=s_j$. Let M_0^* denote the 2-sphere which is the union of X_j and the sub-disk of D_r bounded by s_j. That $C_0\subset\operatorname{Int} M_0^*$ is readily established, for $C_0\subset\operatorname{Ext} M_0$ leads to a contradiction. Similarly a 2-sphere M_1^* with $C_1\subset\operatorname{Int} M_1$ is formed from M and D_s.

Now let $\eta>0$ be chosen less than $\epsilon/2$ and so that $S(C_0,\eta)\subset\operatorname{Int} M_0^*$, $S(C_1,\eta)\subset\operatorname{Int} M_1^*$, and $S(C,\eta)\cap(M_0^*\cup M_1^*)=S(C,\eta)\cap(D_r\cup D_s)$. Choose from the sequence $\{M_i\}$ a 2-sphere N with $N\subset S(C,\eta)$ such that $C\subset\operatorname{Int} N$ and N meets each of the sets A and B in a single point. As before, it may be assumed that each component of $N\cap(D_r\cup D_s)=N\cap(M_0^*\cup M_1^*)$ is a simple closed curve separating $N\cap A$ and $N\cap B$ on N. These simple closed curves $s_1, s_2, \cdots, s_n$ bound sub-disks $X_1, X_2, \cdots, X_n$ of $N\backslash B$ and since the s_i are disjoint and each X_i contains $N\cap A$, it may be assumed that $X_1\subset X_2\subset\cdots\subset X_n$. Some of the s_i are on D_r and some are on D_s, so an index j can be found such that s_j lies on one of these two sets (say on D_r) while s_{j+1} lies on the other. The pair s_j, s_{j+1} bound an annular ring R_j on N and s_j bounds a disk D_{rj} on D_r while s_{j+1} bounds a disk D_{sj} on D_s. Let $K = D_{rj} \cup R_j \cup D_{sj}$. That K is a topological 2-sphere which is locally polyhedral modulo C follows from the construction, as does the fact that $K \cap C = R \cup S$ where both R and S are in T. Thus $\mathfrak{K}$ is in $\mathfrak{P}(T, \epsilon)$ provided $T \subset \text{Int } K$ and $K \subset S(T, \epsilon)$.

To see that the latter holds, it is noted first that $D_{rj} \cup D_{sj} \subset D_r \cup D_s \subset S(T, \epsilon/2)$, so $K \subset S(T, \epsilon)$ if $R_j \subset S(T, \epsilon)$. Suppose there is a point p of R_j such that $d(p, T) \geqq \epsilon$. Since $R_j \subset N \subset S(C, \eta)$ and $\eta < \epsilon/2$, there is a point q of C such that $d(p, q) < \epsilon/2$. That $q \in C_2$ cannot be, for then $d(p, T) \leqq d(p, q) + d(q, T) < \epsilon$, so $q \in C_i$ for $i = 0$ or $i = 1$. But $S(C_i, \eta) \subset \text{Int } M_i^*$ so this requires $p \in \text{Int } M_i^*$ for one choice of i, say $i = 0$. The choice of M_0^* and M_1^* is seen to assure that $M_0^* \subset \text{Ext } M_1^*$ and $M_1^* \subset \text{Ext } M_0^*$, and since $R_j \cap (M_0^* \cup M_1^*) = s_j \cup s_{j+1} = \partial R_j$, $R_j \backslash (s_j \cup s_{j+1})$ lies in $(\text{Ext } M_0^*) \cap (\text{Ext } M_1^*)$. Hence $R_j \cap \text{Int } M_0^* = \square$ and $p \in \text{Int } M_0^*$ is a contradiction, proving that $K \subset S(T, \epsilon)$.

It follows from the arguments above that Int K and Int M_i^* are disjoint for $i = 0$ and $i = 1$. Hence the components C_0' and C_1' of $C \backslash K$ determined by C_0 and C_1 are in Ext K. If $T \subset \text{Ext } K$ then the third component C_T of $C \backslash K$ is also in Ext K and a contradiction is reached just as in the proof of Lemma 6.1.

Thus since ϵ and T were arbitrary and $K \in \mathfrak{P}(T, \epsilon)$ has been found, it follows that C has property $\mathcal{P}$.

9. **Conclusion.** Theorems 8.2 and 8.3 combine to give the following result.

9.1. THEOREM. *If C is a k-cell in R^3 for $k = 1$, 2, or 3, and has any two of the following three properties, then it also has the third.*

1. *Property $\mathcal{P}$.*
2. *The uniform disk property.*
3. *The strong enclosure property.*

9.2. THEOREM. *If C is a 1-cell in R^3 with property $\mathcal{P}$, then C has the strong enclosure property.*

If the word strong is deleted here, this becomes a restatement of Theorem 1 of Harrold [6]. The proof given by Harrold, together with the proof of sufficiency in Theorem 8.2, establish 9.2 as stated.

9.3. COROLLARY. *If C is a 1-cell with property $\mathcal{P}$, then C has the uniform disk property.*

Example 1.1 of Fox-Artin [4] is a 1-cell which can be shown to have the uniform disk property but not the enclosure property, and hence of course, not property $\mathcal{P}$. Whether or not a 2-cell or a 3-cell with property $\mathcal{P}$ can fail to have the enclosure property and/or the uniform disk property is an unanswered question.

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